Toric Nash Blowups

Counterexample to Nash-Semple conjecture

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Overview

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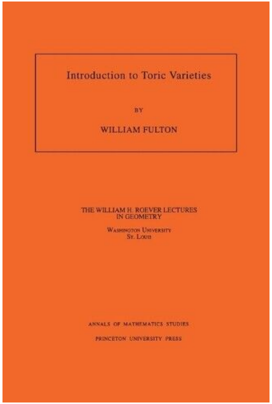
Toric varieties have provided a remarkably fertile testing ground for general theories. Fulton.



Resolution of Singularities

Main goal

Given a variety X, find a variety X' and a projective morphism $f : X' \to X$ such that X' is **smooth** and f is **birational**.

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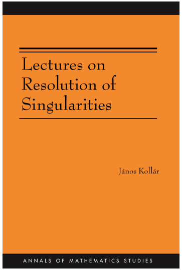
Hironaka (1964) proved constructively that resolutions always exists in charcteristic zero. Positive characteristic is much more subtle.



Tool I: blowups

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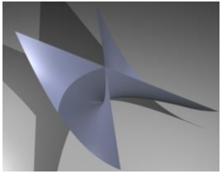
Critical Example

Whitney's umbrella. $V(x^2 - y^2 z) \subseteq \mathrm{Spec}(k[x, y, z])$.

Blowing up the origin results in an isomorphic copy of the umbrella in one of the charts.



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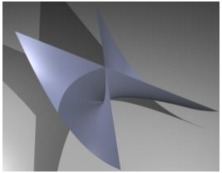

Wrong strategy

Just keep blowing singular points does not always work.

Tool II: Normalization

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- Curves are resolved by normalization.
- A normal surface can be resolved by repeatedly blowing up a singular point and immediatly normalizing.

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The dream is to eliminate choices.

Toric Varieties

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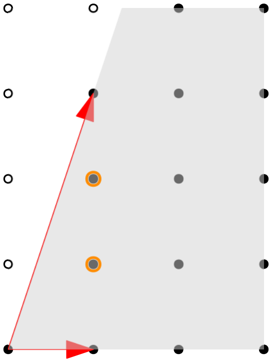

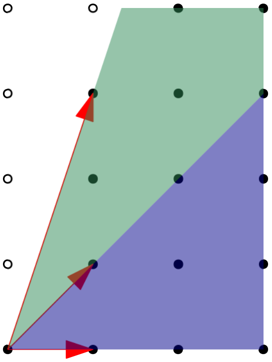

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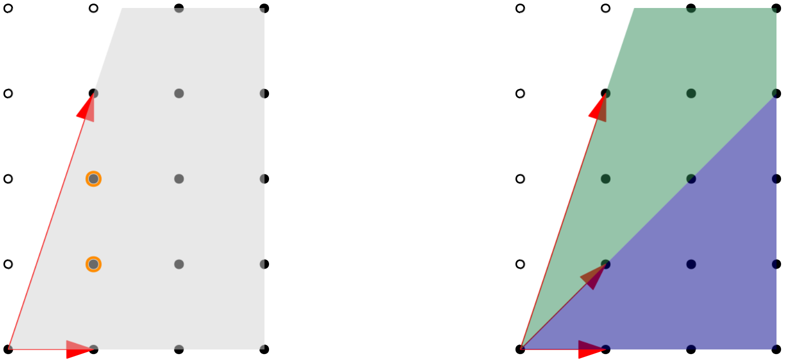

We can **choose** any of the orange points to subdivide. Not necessarily blowups of subvarieties.

Toric blowups

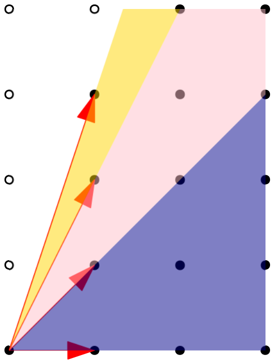

Figure: Smooth subdivision of the previous cone.

Nash blowup

Main motivation

Have a canonical operation, no choices involved.

Definitions

Gauss map

$$\Phi\colon X \setminus \mathrm{Sing}(X) \to \mathrm{Grass}(d, n) \quad \text{defined by} \quad x \mapsto T_x X\,,$$

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Examples

The curve $y^2 - x^3$ in characteristic 3 is isomorphic to its Nash blowup.

First properties

These are nontrivial operations.

Nobile 74

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Duarte-Nunez 22

If X is normal, then the Nash blowup of X is isomorphic to X if and only if X is smooth.

First results

Spivakovsky 90

Iterated normalized Nash blowups eventually resolve singularities for complex surfaces.

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Natural question

Is the same true for arbitrary dimensions?

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Spoiler Alert

No.

Main ingredient

Hilbert Basis

The Hilbert basis of a (rational) cone is the **minimal** set of generators of the semigroup formed by the lattice points in the cone.

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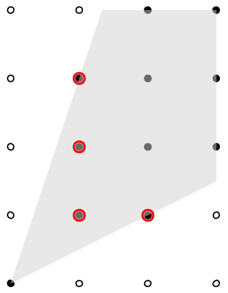
The cone κ generated by $(2,1),(1,3)$ induces a semigroup $\kappa \cap \mathbb{Z}^2$. The minimal (or **irreducible** elements are highlited).



Toric Nash blowup

Following the work of Gonzalez-Teissier, Gonzalez Springer, and Duarte-Jeffreis-Nunez, we have the following description.

Recipe for characteristic 0

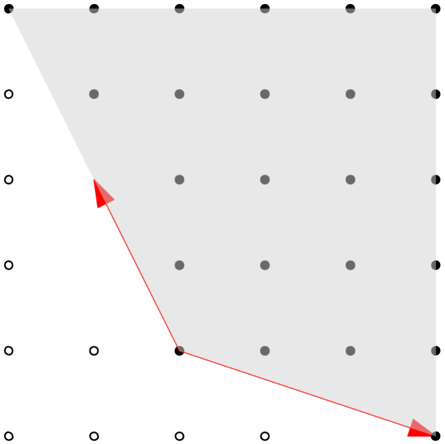
The **normalized** Nash blowup of a 2-cone σ can be constructed as follows:

1. Find $\sigma^\vee$.
2. Compute the Hilbert Basis of it.
3. Add all pairs of linearly independent elements of the Hilbert Basis to obtain S.
4. Consider the polyhedron $S + \sigma^\vee$.
5. Its normal fan subdivides σ.

The toric variety of the last fan is the Nash blowup of σ.

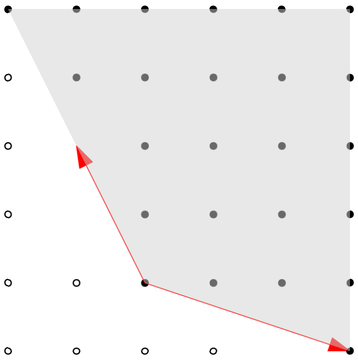
Extended Example

Consider the cone σ.



Extended Example

We compute its dual cone $\sigma^\vee$. This is a basic operation (for example we can use `SageMath`).



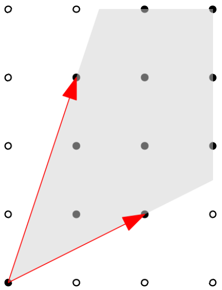


Figure: Dual to the left.

Extended Example

We find its Hilbert Basis (using software like `normaliz`).

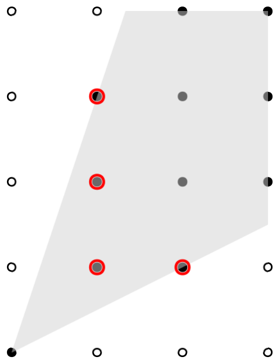
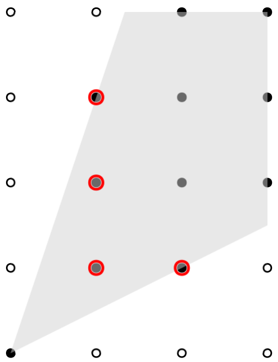


Figure: Hilbert basis

Extended Example

We add lin. ind. pairs and add the dual cone to obtain a polyhedron.



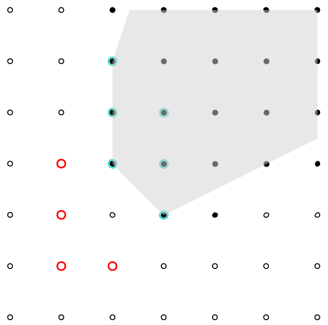
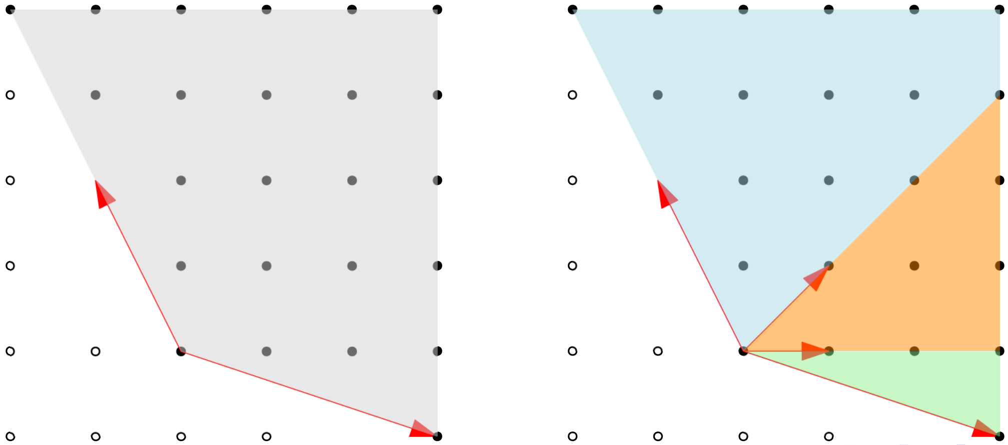


Figure: Auxilliary polyhedron.

Extended Example



Revisiting recipe

It turns out that the recipe extends to positive characteristic.

Recipe char free

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- Everything is computable.
- We can work only with the dual.
- The third part can be checked with a determinant.

Directed graph

We put a directed edge between a cone σ and τ if τ is **isomorphic** to one of the cones resulting in the Nash blowup of σ.

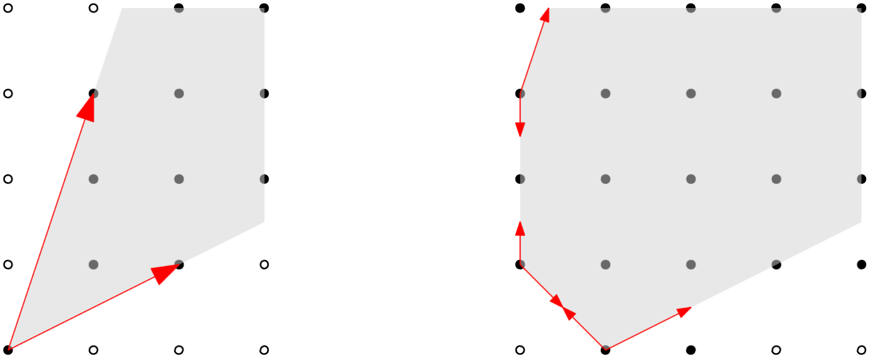

Figure: A cone and its three children.

Conjecture

The **Nash-Semple** conjecture in this terminology is:

Conjecture

Every directed path the graph ends up in the (unique up to isomorphism) smooth cone.

Computations

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Reeves cones. Cones generated by columns of the following matrix

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Dimension 4

Counterexample

The cone $R(4,5)$ does **not** gets resolved in char 0 as it enters a loop.

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Counter in char $\neq 2, 3$

The cone generated by the columns of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix},$$

is isomorphic to one of the cones resulting in its normalized Nash blowup. In fact, the same is true with Nash blowup (w/o normalizing).

Positive characteristic

Counterexample

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The loops found here are different from the above.

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In characteristic zero we have found

- in dimension 4 a unique loop and it has size 1.
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Nash-Semple revisited

Normalized Nash blowups resolve singularities in dimension 3.

The End